The Cardinality of a Union

There is a time when we combine ∪, ∩, and the cardinality of a set. If we have two sets, A and B, and we know the value of $n(A)$ and $n(B)$, then we might want to know the $n(A \cup B)$. Consider a first example: $A=\{1,3,5,7\}$ and $B=\{6,8,10\}$, then $n(A)=4$ and $n(B)=3$. $A \cup B=\{1,3,5,6,7,8,10\}$ so $n(A \cup B)=7$. In this case the cardinality of the union of two sets was the sum of the cardinalities of those two sets. Before we jump to such a conclusion, let us try using a different sets. We will use $D=\{3,4,5\}$. Now $n(D)=3$ and the $n(A)=4$ but $A \cup D=\{1,3,4,5,7\}$ which means that the $n(A \cup D)=5$, and 5 is not the sum of 3 and 4. Why is $n(A \cup D)$ less than $n(A)+n(D)$?

$A \cup D$ is made up of the elements of A merged with the elements of D. However, when we combine the two sets, if there are any elements common to the two sets then we only use those elements once in the union. We have 4 elements in A and 3 elements in D but we have two elements that are in both sets, namely 3 and 5. Therefore, if we look at $n(A)+n(D)$ we are counting those two common elements twice in that sum. Of course, $A \cap D=\{3,5\}$ is the set of common elements and $n(A \cap D)=2$, giving us the number of elements in common. This gives us the equation $n(A \cup D)=n(A)+n(D)-n(A \cap D)$.

Returning to our first example $A \cap B=\varnothing$ and $n(A \cap B)=0$ so we still have $n(A \cup B)=n(A)+n(B)-n(A \cap B)$, or as numbers $7=4+3-0$.

This is a general rule of counting elements in sets. For any two sets R and S

$n(R \cup S)=n(R)+n(S)-n(R \cap S)$